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Light-cone gauge superstring field theory and dimensional regularization

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Abstract: We propose a dimensional regularization scheme in the light-cone gauge NSR superstring field theory to regularize the divergences caused by the colliding supercurrents inserted at the interaction points. We study the tree amplitudes and show that our scheme actually regularize the divergences. We examine the four-point amplitudes for the NS-NS closed strings and find that the results in the first-quantized theory are reproduced without any counterterms.

Keywords: String Field Theory, Superstrings and Heterotic Strings, Renormalization Regularization and Renormalons

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Contents

1 Introduction

In the light-cone gauge closed NSR superstring field theory there exists the joining-splitting type cubic interaction. In order that the theory should be Lorentz invariant, the transverse supercurrent must be inserted at the interaction point [1, 2]. In calculating the amplitudes perturbatively, these supercurrents give rise to unwanted divergences when they get close to each other. Since the Green-Schwarz formulation is equivalent to the NSR formulation in the light-cone gauge [3], the same difficulty exists in the light-cone gauge string field theory of Green-Schwarz strings [4, 5].[1]

In order to make these theories well-defined even classically, we have to regularize these divergences and add counterterms to cancel them. The results from the supersheet approach [7, 8] imply that the divergences are formally from the integration of total derivative terms over the moduli space. Therefore, by choosing a good regularization, we may be able to tame the divergences. Usually in the literature, point splitting type regularizations are employed. In the field theory of point particles, the dimensional regularization is very powerful. In this paper, we therefore pursue this regularization in the light-cone gauge NSR superstring field theory to regularize the divergences of the amplitudes originating in the colliding supercurrents.

In string theory, dimensional regularization can be implemented by shifting the number of space-time dimensions, or shifting the central charge of the conformal field theory on the worldsheet. In this paper, we consider the worldsheet CFT which consists of the ordinary

[1] Similar problems in covariant superstring field theories are studied in [6].

transverse coordinate fields and a superconformal field theory with Virasoro central charge $c^{\rm ext}$. It is possible to construct light-cone gauge string field theory with such worldsheet CFT, although it is not Lorentz invariant. We study the tree amplitudes and show that this procedure actually regularizes the above mentioned divergences. The question is what kind of counterterms are needed to cancel the divergences to obtain the amplitudes which coincide with the ones derived by using the first-quantized formalism. In this paper, we examine the four-point tree amplitudes for the NS-NS closed strings and show that no counterterms are needed in this case.

The organization of this paper is as follows. In the next section, we study the N-string tree amplitudes of the light-cone gauge closed string field theory for NSR superstrings with extra CFT, and see that the tree amplitudes become well-defined by making the central charge $c^{\rm ext}$ of the CFT on the worldsheet largely negative. Therefore we can define the amplitudes for such $c^{\rm ext}$ and analytically continue $c^{\rm ext}$ to $c^{\rm ext} = 0$. In section 3, in order to compare the results with those of the first-quantized formalism, we rewrite the NS-NS tree amplitudes by introducing the longitudinal coordinates and the ghosts. For $c^{\rm ext} = 0$, this was done in ref. [8] to show the equivalence between the light-cone gauge amplitudes and the covariant ones. We essentially follow their procedure in the component formalism. In section 4, we show that the four-point tree amplitudes for the NS-NS closed strings are finite in our regularization scheme and they coincide with the results in the first-quantized formalism. Therefore no four-string contact interaction terms are needed as counterterms. Section 5 is devoted to discussions. In appendix A, we summarize our convention for the light-cone gauge string field theory. In appendix B, the Mandelstam mapping is given. In appendix C, a derivation of the oscillator independent part of the tree interaction vertex for N strings is presented.

2 Light-cone gauge superstring amplitude

Let us consider the light-cone gauge closed string field theory of NSR superstrings whose worldsheet theory consists of the usual CFT for the NSR superstring in the ten dimensional space-time and a superconformal field theory with central charge $c^{\rm ext}$. We will refer to the latter as the extra CFT or extra sector. Even with $c^{\rm ext} \neq 0$, one can define the light-cone gauge string field theory using the super Virasoro operators, although the ten dimensional Lorentz invariance is broken. We calculate the amplitudes perturbatively, and the extra sector part in every external state of the amplitudes is always taken to be the conformal vacuum. In this paper, we restrict ourselves to the tree amplitudes. Since the relevant property of the extra CFT is only its central charge in this situation, we do not specify the details of the extra CFT. While we concentrate on the case that all the external strings belong to the NS-NS sector, it is not difficult to extend our calculation into the case where the Ramond strings are involved.

The N-point NS-NS tree amplitudes in the light-cone gauge string field theory are calculated from the action defined in appendix A. They are given as integrals over the moduli space,

$$\mathcal{A} = (ig)^{N-2} C_N \int \left(\prod_I \frac{d^2\mathcal{T}_I}{4\pi} \right) F(\mathcal{T}_I, \bar{\mathcal{T}}_I)\,, \tag{2.1}$$

where g denotes the string coupling, C_N is the symmetric factor, $\mathcal{T}_I$ are the $N-3$ complex moduli parameters of the string diagram defined as

$$\mathcal{T}_I = \rho(z_{I+1}) - \rho(z_I) = T_I + i\alpha_m\theta_I\,, \qquad \alpha_m > 0\,, \tag{2.2}$$

and

$$\int \frac{d^2\mathcal{T}_I}{4\pi} = -i\alpha_m \int_0^\infty dT_I \int_0^{2\pi} \frac{d\theta_I}{2\pi}\,. \tag{2.3}$$

Here $\rho(z)$ is the Mandelstam mapping [9] given in eq. (B.2), z_I denote the coordinates of the $N-2$ interaction points on the complex z-plane defined in appendix B and α_m denote the string-length parameters of the intermediate strings. The propagator for the intermediate strings can be written as

$$-\,i \int \frac{d^2\mathcal{T}_I}{4\pi} e^{-\frac{\mathcal{T}_I}{\alpha_m}(L_0^{\mathrm{LC}}-\frac{c}{24})-\frac{\bar{\mathcal{T}}_I}{\alpha_m}(\tilde{L}_0^{\mathrm{LC}}-\frac{c}{24})} |R(m,m')\rangle \frac{1}{-i\alpha_m}\,, \qquad \text{with } \alpha_m > 0\,, \tag{2.4}$$

where c is the total central charge of this system,

$$c = c^{(X,\psi)\mathrm{LC}} + c^{\mathrm{ext}}\,, \qquad c^{(X,\psi)\mathrm{LC}} = 12\,. \tag{2.5}$$

Connecting the three-string vertices using the propagator (2.4), we obtain the N-string interaction vertex $\langle V_N|$, which takes the form

$$\langle V_N| = 4\pi\delta\left(\sum_{r=1}^N \alpha_r\right) \langle V_N^{\mathrm{LPP}}| e^{-\Gamma^{\mathrm{LC}}} \prod_I \left(\partial^2\rho(z_I)\bar{\partial}^2\bar{\rho}(\bar{z}_I)\right)^{-\frac{3}{4}} T_F^{\mathrm{LC}}(z_I)\tilde{T}_F^{\mathrm{LC}}(\bar{z}_I)\,. \tag{2.6}$$

Here $\langle V_N^{\mathrm{LPP}}|$ is the LPP vertex that is determined by the correlation functions of the worldsheet theory through the prescription of LeClair, Peskin and Preitschopf [10]. We take the normalization of this vertex as

$$\int \left(\prod_{r=1}^N \frac{d^8 p_r^i}{(2\pi)^8}\right) \langle V_N^{\mathrm{LPP}}|0\rangle_1 \left(\prod_{r=2}^N |0\rangle_r (2\pi)^8 \delta^8(p_r^i)\right) = 1\,. \tag{2.7}$$

T_F^{LC} denotes the transverse supercurrent of the worldsheet theory given in eq. (A.5). $e^{-\Gamma^{\mathrm{LC}}}$ is the value of the N-point amplitude without any supercurrent insertions when all the external states are taken to be the conformal vacua.

The overlap of the LPP vertex with external states can be related to the correlation functions on the complex z-plane via the Mandelstam mapping $\rho(z)$. Therefore, the integrand $F(\mathcal{T}_I, \bar{\mathcal{T}}_I)$ of the amplitude (2.1) can be expressed by the path integral,[2]

$$
\begin{aligned}
F(\mathcal{T}_I,\bar{\mathcal{T}}_I) \sim{}& (4\pi)\delta\left(\sum_{r=1}^{N}\alpha_r\right)(2\pi)\delta\left(\sum_{r=1}^{N}p_r^-\right)\\
&\times\int[dX^i d\psi^i d\phi^{\mathrm{ext}}]\,e^{-S^{\mathrm{LC}}-\Gamma^{\mathrm{LC}}}\prod_{r=1}^{N}\left[\alpha_r V_r^{\mathrm{LC}}(w_r=0,\bar{w}_r=0)e^{-p_r^-\tau_0^{(r)}}\right]\\
&\times\prod_I\left[\left(\partial^2\rho(z_I)\bar{\partial}^2\bar{\rho}(\bar{z}_I)\right)^{-\frac{3}{4}}T_F^{\mathrm{LC}}(z_I)\tilde{T}_F^{\mathrm{LC}}(\bar{z}_I)\right]\,. \qquad (2.8)
\end{aligned}
$$

Here $\tau_0^{(r)}$ is defined in eq. (B.3), and w_r is the coordinate of the unit disk of the r-th external string given in appendix B, the origin of which corresponds to the puncture at $z = Z_r$ on the z-plane. ϕ^{ext} denotes the fields in the extra sector and S^{LC} is the action of the worldsheet theory of the light-cone gauge NSR superstring containing the extra sector. $V_r^{\mathrm{LC}}(w_r, \bar{w}_r)$ denotes the vertex operator of the r-th string corresponding to the string state

$$
\epsilon^{(r)}_{\{ij\}}\alpha^{i_1(r)}_{-n_1}\cdots\tilde{\alpha}^{\tilde{\imath}_1(r)}_{-\tilde{n}_1}\cdots\psi^{j_1(r)}_{-s_1}\cdots\tilde{\psi}^{\tilde{\jmath}_1(r)}_{-\tilde{s}_1}\cdots|p_r^i\rangle_r\,, \qquad (2.9)
$$

namely

$$
\begin{aligned}
V_r^{\mathrm{LC}}(w_r,\bar{w}_r) ={}& \epsilon^{(r)}_{\{ij\}}\oint_{w_r}\frac{dw_{n_1}}{2\pi i}\frac{i\partial X^{i_1(r)}(w_{n_1})}{(w_{n_1})^{n_1}}\cdots\oint_{\bar{w}_r}\frac{d\bar{w}_{\tilde{n}_1}}{2\pi i}\frac{i\bar{\partial}X^{\tilde{\imath}_1(r)}(\bar{w}_{\tilde{n}_1})}{(\bar{w}_{\tilde{n}_1})^{\tilde{n}_1}}\cdots\\
&\times\oint_{w_r}\frac{dw_{s_1}}{2\pi i}\frac{\psi^{j_1(r)}(w_{s_1})}{(w_{s_1})^{s_1+\frac{1}{2}}}\cdots\oint_{\bar{w}_r}\frac{d\bar{w}_{\tilde{s}_1}}{2\pi i}\frac{\tilde{\psi}^{\tilde{\jmath}_1(r)}(\bar{w}_{\tilde{s}_1})}{(\bar{w}_{\tilde{s}_1})^{\tilde{s}_1+\frac{1}{2}}}\cdots e^{ip_r^iX^{i(r)}}(w_r,\bar{w}_r)\,, \qquad (2.10)
\end{aligned}
$$

where $\epsilon^{(r)}_{\{ij\}}$ is a shorthand notation of the polarization tensor $\epsilon^{(r)}_{i_1\cdots j_1\cdots\tilde{\imath}_1\cdots\tilde{\jmath}_1\cdots}(p_r^i)$. For later use, we introduce the mode numbers N_r and $\tilde{N}_r$ of this vertex operator defined as

$$
N_r = \sum_l n_l + \sum_l s_l\,, \qquad \tilde{N}_r = \sum_l \tilde{n}_l + \sum_l \tilde{s}_l\,. \qquad (2.11)
$$

The level-matching condition and the on-shell condition[3] require that

$$
N_r = \tilde{N}_r\,, \qquad p_r^\mu p_{\mu r} + N_r + \tilde{N}_r - 1 = 0\,. \qquad (2.12)
$$

We note that the factor α_r in front of each vertex operator V_r^{LC} in the integrand in eq. (2.8) comes from that contained in the measure dr defined in eq. (A.2).

$e^{-\Gamma^{\mathrm{LC}}}$ can be obtained by evaluating the Liouville action for the flat metric on the light-cone diagram [11] as briefly illustrated in appendix C, and we have

$$
e^{-\Gamma^{\mathrm{LC}}} = \mathrm{sgn}\left(\prod_{r=1}^{N}\alpha_r\right)\left|\prod_{r=1}^{N}\alpha_r\right|^{-\frac{c}{12}}\left|\sum_{s=1}^{N}\alpha_s Z_s\right|^{\frac{c}{6}}e^{-\frac{c}{12}\sum_{r=1}^{N}\mathrm{Re}\,\bar{N}_{00}^{rr}}\prod_I\left|\partial^2\rho(z_I)\right|^{-\frac{c}{24}}\,. \qquad (2.13)
$$

[2] We are not interested in overall numerical factors of the amplitudes. The symbol $\sim$ will be used to indicate the equality up to a constant factor.

[3] One may consider that the on-shell condition should be $p_r^\mu p_{\mu r} + N_r + \tilde{N}_r - \frac{c}{12} = 0$ rather than eq. (2.12). The difference between these two on-shell conditions has no effect in the limit $c^{\mathrm{ext}} \to 0$.

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Here $\bar{N}^{rr}_{00}$ is a Neumann coefficient given by

$$\bar{N}^{rr}_{00} = -\sum_{s\neq r} \frac{\alpha_s}{\alpha_r} \ln(Z_r - Z_s) + \frac{\tau_0^{(r)} + i\beta_r}{\alpha_r}\,, \tag{2.14}$$

and β_r is defined in eq. (B.1).

Combining eqs. (2.8) and (2.13), we obtain

$$\begin{aligned} F(\mathcal{T}_I, \bar{\mathcal{T}}_I) \sim\ & (4\pi)\delta\left(\sum_{r=1}^{N} \alpha_r\right)(2\pi)\delta\left(\sum_{r=1}^{N} p_r^-\right)\int [dX^i d\psi^i d\phi^{\mathrm{ext}}] e^{-S^{\mathrm{LC}}}\, |\mathcal{D}|^{2c^{\mathrm{ext}}} \left|\sum_{s=1}^{N} \alpha_s Z_s\right|^2 \\ & \times \prod_{r=1}^{N}\left[V_r^{\mathrm{LC}}(w_r = 0, \bar{w}_r = 0) e^{-p_r^- \tau_0^{(r)}} e^{-\mathrm{Re}\,\bar{N}^{rr}_{00}}\right] \\ & \times \prod_I \left[\left(\partial^2\rho(z_I)\bar{\partial}^2\bar{\rho}(\bar{z}_I)\right)^{-1} T_F^{\mathrm{LC}}(z_I)\tilde{T}_F^{\mathrm{LC}}(\bar{z}_I)\right], \end{aligned} \tag{2.15}$$

where $\mathcal{D}$ is defined as

$$\mathcal{D} = \left|\prod_{r=1}^{N} \alpha_r\right|^{-\frac{1}{24}} \left(\sum_{s=1}^{N} \alpha_s Z_s\right)^{\frac{1}{12}} e^{-\frac{1}{24}\sum_{r=1}^{N} \bar{N}^{rr}_{00}} \left(\prod_I \partial^2\rho(z_I)\right)^{-\frac{1}{48}}. \tag{2.16}$$

In general, $F(\mathcal{T}_I, \bar{\mathcal{T}}_I)$ is singular in the limit $z_I - z_J \to 0$. Taking eq. (B.4) into account, we find that if c^{ext} is taken to be a sufficiently large negative value as a regularization, the amplitude (2.1) becomes non-singular as $z_I - z_J \to 0$. Other singularities can be dealt with by the analytic continuation of the external momenta p_r. Therefore one can define the integral in eq. (2.1) for such c^{ext} and analytically continue c^{ext} to $c^{\mathrm{ext}} = 0$. Thus one obtains the dimensionally regularized amplitudes.

While we have studied only the NS-NS closed strings, it is easy to find that the regularization scheme mentioned above works for the amplitudes involving the Ramond strings as well.

3 Relation to covariant formulation

We would like to compare the NS-NS tree amplitudes defined in the last section with those calculated using the first-quantized formalism. In order to do so, it is convenient to recast the integrand $F(\mathcal{T}_I, \bar{\mathcal{T}}_I)$ in eq. (2.15) into a "covariant" form, by introducing the longitudinal coordinates and the ghosts. For $c^{\mathrm{ext}} = 0$, such a procedure was given in ref. [8] using the superspace formalism. Here we would like to follow their procedure in the component language. Since we are dealing with $c^{\mathrm{ext}} \neq 0$ case, we cannot obtain Lorentz covariant amplitudes by doing so. However, we can obtain the amplitudes which can be compared with the covariant ones in the limit $c^{\mathrm{ext}} \to 0$.

Ghosts. Let us first introduce the ghost fields b, c, β, γ. We bosonize the $\beta\gamma$-ghosts in the usual way [12] as

$$\beta(z) = e^{-\phi}\partial\xi(z)\,, \quad \gamma(z) = \eta e^{\phi}(z)\,. \tag{3.1}$$

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On the complex plane in which we are interested here, we have the following identity,

$$\begin{aligned}
&\int [dbdcd\beta d\gamma]\, e^{-S_{\rm gh}} \left(\lim_{z,\bar{z}\to\infty} \frac{1}{|z|^4} c(z)\tilde{c}(\bar{z}) \right) \\
&\qquad \times \prod_I \left[b(z_I)\tilde{b}(\bar{z}_I) e^{\phi}(z_I) e^{\tilde{\phi}}(\bar{z}_I) \right] \prod_{r=1}^N \left[c(Z_r)\tilde{c}(\bar{Z}_r) e^{-\phi}(Z_r) e^{-\tilde{\phi}}(\bar{Z}_r) \right] \\
&\quad \sim \frac{\prod_{r<s}|Z_r - Z_s|^2 \prod_{I<J}|z_I - z_J|^2}{\prod_{I,r}|z_I - Z_r|^2} \times \frac{\prod_{I,r}|z_I - Z_r|^2}{\prod_{r<s}|Z_r - Z_s|^2 \prod_{I<J}|z_I - z_J|^2} = 1\,, \qquad (3.2)
\end{aligned}$$

where $S_{\rm gh}$ denotes the worldsheet action for $(b,c;\beta,\gamma)$. The two factors in the third line in this equation represent the contributions from the (b,c) and the (β,γ) sectors, respectively. One can notice that the ghost number of the operators inserted in the path integral (3.2) is the one that makes this correlation function nonvanishing.

Since the left hand side of eq. (3.2) is just a constant, we can introduce the ghost sector by multiplying the right hand side of eq. (2.15) by this path integral, without changing $F(\mathcal{T}_I, \bar{\mathcal{T}}_I)$.

Longitudinal coordinates. Next, let us consider the longitudinal coordinates $X^\pm, \psi^\pm$. For the path integral of these fields, we have

$$\begin{aligned}
&\int [dX^\pm d\psi^\pm] e^{-S_\pm} \prod_{r=1}^N V_r^{\rm DDF}(Z_r, \bar{Z}_r) \\
&\qquad \sim (2\pi)^2 \delta\left(\sum_{r=1}^N p_r^+ \right) \delta\left(\sum_{r=1}^N p_r^- \right) \prod_{r=1}^N \left[V_r^{\rm LC}(w_r = 0, \bar{w}_r = 0) e^{-p_r^- \tau_0^{(r)} - {\rm Re}\,\bar{N}_{00}^{rr}} \right]\,. \qquad (3.3)
\end{aligned}$$

Here $S_\pm$ denotes the worldsheet action for $(X^\pm, \psi^\pm)$. $V_r^{\rm DDF}$ is the vertex operator for the DDF state corresponding to $V_r^{\rm LC}$, the explicit form of which will be given below. We can introduce the longitudinal coordinates by substituting eq. (3.3) into the right hand side of eq. (2.15).

In the following, we will prove eq. (3.3) by performing the path integral on the left hand side. Before doing this, let us illustrate what $V_r^{\rm DDF}(Z_r, \bar{Z}_r)$ is. The DDF state corresponding to the light-cone gauge string state (2.9) is given by

$$\epsilon_{\{ij\}}^{(r)} A_{-n_1}^{i_1(r)} \cdots \tilde{A}_{-\tilde{n}_1}^{\tilde{i}_1(r)} \cdots B_{-s_1}^{j_1(r)} \cdots \tilde{B}_{-\tilde{s}_1}^{\tilde{j}_1(r)} \cdots \left| p_r + \frac{1}{2}\left(N_r + \tilde{N}_r \right) k_r \right\rangle_r\,, \qquad (3.4)$$

where N_r, $\tilde{N}_r$ are the mode numbers given in eq. (2.11), k_r is the 10-momentum with

$$k_r^- = -\frac{1}{p_r^+}\,, \qquad k_r^+ = k_r^i = 0\,, \qquad (3.5)$$

the operators $A_{-n}^{i(r)}$ and $B_{-s}^{i(r)}$ are defined as

$$\begin{aligned}
A_{-n}^{i(r)} &= \oint_{Z_r} \frac{dz_n^{(r)}}{2\pi i} \left(i\partial X^i + \frac{n}{p_r^+} \psi^i \psi^+ \right) e^{-i\frac{n}{p_r^+} X_L^+}(z_n^{(r)})\,, \\
B_{-s}^{i(r)} &= \oint_{Z_r} \frac{dz_s^{(r)}}{2\pi i} \left(\psi^i - \partial X^i \frac{\psi^+}{\partial X^+} - \frac{1}{2}\psi^i \frac{\psi^+ \partial \psi^+}{(\partial X^+)^2} \right) \left(\frac{i\partial X^+}{p_r^+} \right)^{1/2} e^{-i\frac{s}{p_r^+} X_L^+}(z_s^{(r)})\,, \quad (3.6)
\end{aligned}$$

and similarly for their anti-holomorphic counterparts $\tilde{A}^{\tilde{i}(r)}_{-\tilde{n}}$ and $\tilde{B}^{\tilde{i}(r)}_{-\tilde{s}}$. Here $X^+_L(z)$ and $X^+_R(\bar{z})$ denote the holomorphic and the anti-holomorphic parts of $X^+(z,\bar{z})$. $V^{\rm DDF}_r(Z_r,\bar{Z}_r)$ is the vertex operator for the DDF state (3.4), given by

$$V^{\rm DDF}_r(Z_r,\bar{Z}_r) = \epsilon^{(r)}_{\{ij\}} A^{i_1(r)}_{-n_1}\cdots \tilde{A}^{\tilde{i}_1(r)}_{-\tilde{n}_1}\cdots B^{j_1(r)}_{-s_1}\cdots \tilde{B}^{\tilde{j}_1(r)}_{-\tilde{s}_1}\cdots \\ \times e^{ip^i_r X^i}(Z_r,\bar{Z}_r) e^{-ip^+_r X^- - i\left(p^-_r - \frac{N_r+\tilde{N}_r}{2p^+_r}\right)X^+}(Z_r,\bar{Z}_r)\,. \tag{3.7}$$

Since there appears no ψ^- in $V^{\rm DDF}_r(Z_r,\bar{Z}_r)$, the expectation values of the operators involving ψ^+'s vanish and thus integrating over $\psi^\pm$ yields just the partition function, in the path integral (3.3). Since X^- appears only in the form $e^{-ip^+_r X^-}$, it is also easy to carry out the path integral over $X^\pm$ and we obtain

$$\begin{aligned}
&\int [dX^\pm] e^{-S^X_\pm} \prod_{r=1}^N \Bigg[e^{-i\frac{n_1}{p^+_r}X^+_L}(z^{(r)}_{n_1})\cdots \left(\frac{i\partial X^+}{p^+_r}\right)^{1/2} e^{-i\frac{s_1}{p^+_r}X^+_L}(z^{(r)}_{s_1})\cdots \\
&\qquad \times e^{-i\frac{\tilde{n}_1}{p^+_r}X^+_R}(\bar{z}^{(r)}_{\tilde{n}_1})\cdots \left(\frac{i\bar{\partial} X^+}{p^+_r}\right)^{1/2} e^{-i\frac{\tilde{s}_1}{p^+_r}X^+_R}(\bar{z}^{(r)}_{\tilde{s}_1})\cdots e^{-ip^+_r X^- - i\left(p^-_r - \frac{N_r+\tilde{N}_r}{2p^+_r}\right)X^+}(Z_r,\bar{Z}_r)\Bigg] \\
&\sim (2\pi)^2 \delta\left(\sum_{r=1}^N p^+_r\right)\delta\left(\sum_{r=1}^N p^-_r\right) \prod_{r=1}^N \Bigg[\left(w^{(r)}_{n_1}\right)^{-n_1}\cdots \left(\frac{\partial w^{(r)}_{s_1}}{\partial z^{(r)}_{s_1}}\right)^{\frac{1}{2}} \left(w^{(r)}_{s_1}\right)^{-s_1-\frac{1}{2}}\cdots \\
&\qquad \times \left(\bar{w}^{(r)}_{\tilde{n}_1}\right)^{-\tilde{n}_1}\cdots \left(\frac{\partial \bar{w}^{(r)}_{\tilde{s}_1}}{\partial \bar{z}^{(r)}_{\tilde{s}_1}}\right)^{\frac{1}{2}} \left(\bar{w}^{(r)}_{\tilde{s}_1}\right)^{-\tilde{s}_1-\frac{1}{2}}\cdots e^{(p^i_r p^i_r - 1)\,{\rm Re}\,\bar{N}^{rr}_{00}} e^{-p^-_r \tau^{(r)}_0}\Bigg]\,,
\end{aligned} \tag{3.8}$$

where $S^X_\pm$ is the $X^\pm$ part of the worldsheet action $S_\pm$, and $w^{(r)}_n$ denotes the coordinate on the unit disk w_r defined in eq. (B.1) which corresponds to $z^{(r)}_n$. Here we have used the level-matching condition and the on-shell condition (2.12). By using eq. (3.8) and the relation

$$e^{ip^i_r X^i}(Z_r,\bar{Z}_r) = e^{-p^i_r p^i_r\,{\rm Re}\,\bar{N}^{rr}_{00}} e^{ip^i_r X^{i(r)}}(w_r=0,\bar{w}_r=0)\,, \tag{3.9}$$

we obtain eq. (3.3).

With the ghosts and the longitudinal coordinates thus introduced, we have the expression

$$F(\mathcal{T}_I,\bar{\mathcal{T}}_I) \sim \int [dXd\psi dbdcd\beta d\gamma d\phi^{\rm ext}] e^{-S} |\mathcal{D}|^{2c^{\rm ext}} \left(\lim_{z,\bar{z}\to\infty} \frac{1}{|z|^4} c(z)\tilde{c}(\bar{z})\right) \left|\sum_{s=1}^N \alpha_s Z_s\right|^2 \\ \times \prod_{r=1}^N \left[c\tilde{c}e^{-\phi}e^{-\tilde{\phi}} V^{\rm DDF}_r(Z_r,\bar{Z}_r)\right] \prod_I \left[\frac{b(z_I)}{\partial^2\rho(z_I)} \frac{\tilde{b}(\bar{z}_I)}{\bar{\partial}^2\bar{\rho}(\bar{z}_I)} T^{\rm LC}_F e^{\phi}(z_I) \tilde{T}^{\rm LC}_F e^{\tilde{\phi}}(\bar{z}_I)\right], \tag{3.10}$$

where S is the total worldsheet action $S = S^{\rm LC} + S_\pm + S_{\rm gh}$.

Picture changing operator. In the following, we will show that $T^{\rm LC}_F$ in eq. (3.10) can be rewritten by using the picture changing operator in ten dimensions. Let us introduce a fermionic charge Q defined as

$$Q = \frac{1}{2}\oint \frac{dz}{2\pi i} \partial\rho(z)\Big(c(z)\left(2i\partial X^+(z) - \partial\rho(z)\right) + \eta(z)e^{\phi}(z)\psi^+(z)\Big)\,. \tag{3.11}$$

We notice that this charge is nilpotent $Q^2 = 0$, and because of the relations

$$\begin{aligned}\left[Q\,,\,c\tilde{c}e^{-\phi}e^{-\tilde{\phi}}V_r^{\mathrm{DDF}}(Z_r,\bar{Z}_r)\right] = 0\,, \quad &\left[Q\,,\,T_F^{\mathrm{LC}}e^{\phi}(z_I)\right] = 0\,,\\ \left\{Q\,,\,\frac{b(z_I)}{\partial^2\rho(z_I)}\right\} = 0\,, \qquad &\{Q\,,\,c(\infty)\} = 0\,,\end{aligned} \tag{3.12}$$

Q commutes with the operators which appear on the right hand side of eq. (3.10). Therefore, we can replace $T_F^{\mathrm{LC}}e^{\phi}(z_I)$ in eq. (3.10) by

$$\begin{aligned}X'(z_I) &\equiv T_F^{\mathrm{LC}}e^{\phi}(z_I) - \frac{1}{\partial^2\rho(z_I)}\left\{Q\,,\,i\partial X^-\partial\xi(z_I) + \frac{1}{2}\partial b e^{\phi}\psi^-(z_I)\right\}\\ &= \left(T_F^{\mathrm{LC}}e^{\phi} + c\partial\xi + \frac{i}{2}\left(\psi^+\partial X^- + \psi^-\partial X^+\right)e^{\phi} + \frac{1}{4}\partial b\eta e^{2\phi}\right)(z_I)\,,\end{aligned} \tag{3.13}$$

without changing $F(\mathcal{T}_I,\bar{\mathcal{T}}_I)$. $X'(z_I)$ thus defined satisfies

$$b(z_I)X'(z_I) = b(z_I)\left(\{Q_{\mathrm{B}}\,,\,\xi(z_I)\} + T_F^{\mathrm{ext}}e^{\phi}(z_I)\right)\,, \tag{3.14}$$

where Q_{B} denotes the BRST charge of the covariant NSR superstring theory in the ten dimensional space-time, and $\{Q_{\mathrm{B}}\,,\xi(z_I)\}$ is the picture changing operator, which takes the form

$$\{Q_{\mathrm{B}}\,,\,\xi(z)\} = -\frac{1}{2}i\psi^{\mu}\partial X_{\mu}e^{\phi}(z) + c\partial\xi(z) + \frac{1}{4}\partial b\eta e^{2\phi}(z) + \frac{1}{4}b\left(2\partial\eta e^{2\phi} + \eta(\partial e^{2\phi})\right)(z)\,. \tag{3.15}$$

Consequently, we find that $T_F^{\mathrm{LC}}e^{\phi}(z_I)$ in the path integral (3.10) can be replaced by $\{Q_{\mathrm{B}}\,,\,\xi(z_I)\} + T_F^{\mathrm{ext}}e^{\phi}(z_I)$, and we obtain

$$\begin{aligned}F(\mathcal{T}_I,\bar{\mathcal{T}}_I) \sim &\int \left[dXd\psi dbdcd\beta d\gamma d\phi^{\mathrm{ext}}\right]e^{-S}\\ &\times|\mathcal{D}|^{2c^{\mathrm{ext}}}\left(\lim_{z,\bar{z}\to\infty}\frac{1}{|z|^4}c(z)\tilde{c}(\bar{z})\right)\left|\sum_{s=1}^{N}\alpha_s Z_s\right|^2\prod_{r=1}^{N}\left[c\tilde{c}e^{-\phi}e^{-\tilde{\phi}}V_r^{\mathrm{DDF}}(Z_r,\bar{Z}_r)\right]\\ &\times\prod_I\left[\frac{b(z_I)}{\partial^2\rho(z_I)}\frac{\tilde{b}(\bar{z}_I)}{\bar{\partial}^2\bar{\rho}(\bar{z}_I)}\left(\{Q_{\mathrm{B}},\xi(z_I)\}+T_F^{\mathrm{ext}}e^{\phi}(z_I)\right)\left(\{\tilde{Q}_{\mathrm{B}},\tilde{\xi}(\bar{z}_I)\}+\tilde{T}_F^{\mathrm{ext}}e^{\tilde{\phi}}(\bar{z}_I)\right)\right].\end{aligned} \tag{3.16}$$

4 Regularization of the four-point tree amplitudes

In this section, we restrict our attention to the $N = 4$ case. We define the amplitudes for large negative c^{ext}'s and analytically continue them to $c^{\mathrm{ext}} = 0$. We would like to compare our amplitudes with the ones obtained in the first-quantized formalism.

By using the relation

$$\frac{b(z_I)}{\partial^2\rho(z_I)} = \oint_{z_I}\frac{dz}{2\pi i}\frac{b(z)}{\partial\rho(z)}\,, \tag{4.1}$$

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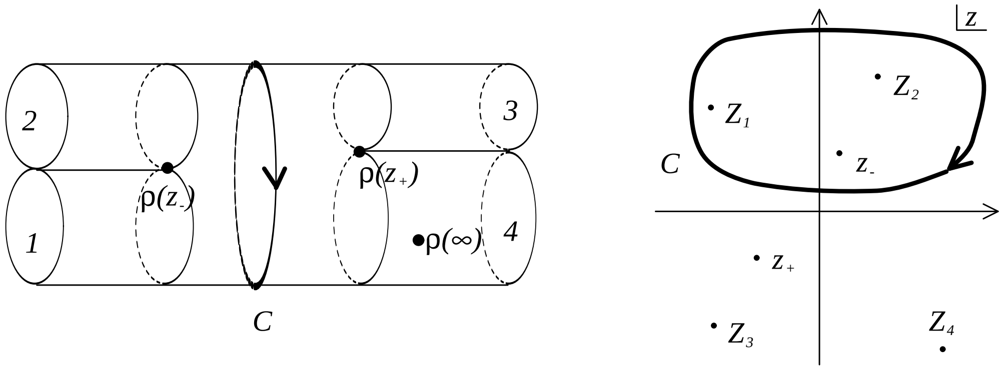

(a) (b)

Figure 1. The contour C on the ρ-plane of the integral (4.2) for the $N = 4$ case is depicted in (a).On the z-plane, it is described in (b).

and deforming the contour of this integral, one can simplify eq. (3.16) for $N = 4$:

$$\begin{aligned} F(\mathcal{T},\bar{\mathcal{T}}) \sim &\int [dXd\psi dbdcd\beta d\gamma d\phi^{\rm ext}]\, e^{-S}\, |\mathcal{D}|^{2c^{\rm ext}} \oint_C \frac{dz}{2\pi i}\frac{b}{\partial\rho} \oint_C \frac{d\bar{z}}{2\pi i}\frac{\tilde{b}}{\bar{\partial}\bar{\rho}} \\ &\times \prod_{r=1}^{4}\left[c(Z_r)\tilde{c}(\bar{Z}_r)e^{-\phi}(Z_r)e^{-\tilde{\phi}}(\bar{Z}_r)V_r^{\rm DDF}(Z_r,\bar{Z}_r)\right] \\ &\times \prod_{I=\pm}\left[\left(\{Q_{\rm B}\,,\,\xi(z_I)\} + T_F^{\rm ext}e^{\phi}(z_I)\right)\left(\{\tilde{Q}_{\rm B}\,,\,\tilde{\xi}(\bar{z}_I)\} + \tilde{T}_F^{\rm ext}e^{\tilde{\phi}}(\bar{z}_I)\right)\right] \,. \end{aligned} \tag{4.2}$$

Here the integration contour C is depicted in figure 1. $z_\pm$ denote the two interaction points defined by $z_- \equiv z_I^{(1)} = z_I^{(2)}$, $z_+ \equiv z_I^{(3)} = z_I^{(4)}$, and thus the one moduli parameter $\mathcal{T}$ becomes $\mathcal{T} = \rho(z_+) - \rho(z_-)$.

Now it is easy to see that the integrand $F(\mathcal{T},\bar{\mathcal{T}})$ is expressed as

$$F(\mathcal{T},\bar{\mathcal{T}}) \sim \left|\mathcal{F}_{\rm 10D}(\mathcal{T}) - \frac{c^{\rm ext}}{6(z_+ - z_-)^3}\mathcal{F}'(\mathcal{T})\right|^2 |\mathcal{D}(\mathcal{T})|^{2c^{\rm ext}}\,, \tag{4.3}$$

where

$$\begin{aligned} \mathcal{F}_{\rm 10D} &= \left\langle \oint_C \frac{dz}{2\pi i}\frac{b}{\partial\rho}\prod_{r=1}^{4} V_r^{(-1)}(Z_r) \prod_{I=\pm}\left[\{Q_{\rm B},\xi(z_I)\}\right]\right\rangle\,, \\ \mathcal{F}' &= \left\langle \oint_C \frac{dz}{2\pi i}\frac{b}{\partial\rho}\prod_{r=1}^{4} V_r^{(-1)}(Z_r) \prod_{I=\pm} e^{\phi}(z_I)\right\rangle\,. \end{aligned} \tag{4.4}$$

Here $\langle\cdots\rangle$ denotes the correlation function of the system $(X^\mu,\psi^\mu;b,c;\beta,\gamma)$, and $V_r^{(-1)}(Z_r)$ is defined by using the holomorphic part $V_{rL}^{\rm DDF}(Z_r)$ of $V_r^{\rm DDF}(Z_r,\bar{Z}_r)$ as

$$V_r^{(-1)}(Z_r) = ce^{-\phi}V_{rL}^{\rm DDF}(Z_r)\,. \tag{4.5}$$

$\mathcal{F}_{10\mathrm{D}}$ is almost the first-quantized result, except that the picture changing operators are inserted at $z_\pm$. These can be moved in the usual manner [12] up to total derivative terms with respect to the moduli parameter. As a result we obtain

$$\mathcal{F}_{10\mathrm{D}} = \mathcal{F}_{\mathrm{fin}} + \partial_{\mathcal{T}}\mathcal{G}\,, \tag{4.6}$$

where

$$\mathcal{F}_{\mathrm{fin}} = \left\langle \oint_C \frac{dz}{2\pi i}\frac{b(z)}{\partial\rho(z)} V_1^{(0)}(Z_1)V_2^{(0)}(Z_2)V_3^{(-1)}(Z_3)V_4^{(-1)}(Z_4)\right\rangle\,, \tag{4.7}$$

and $\mathcal{G}$ can be expressed by a correlation function in the large Hilbert space. $V_r^{(0)}(Z_r)$ is defined as

$$V_r^{(0)}(Z_r) = \oint_{Z_r}\frac{dz}{2\pi i} j_{\mathrm{B}}(z)\xi(Z_r)V_r^{(-1)}(Z_r)\,, \tag{4.8}$$

where $j_{\mathrm{B}}(z)$ denotes the BRST current.

Collecting the results obtained above, we find that the four-point tree amplitudes become

$$\mathcal{A} \sim -g^2C_4\int\frac{d^2\mathcal{T}}{4\pi}\left|\mathcal{F}_{\mathrm{fin}} + \partial_{\mathcal{T}}\mathcal{G} - \frac{c^{\mathrm{ext}}}{6(z_+-z_-)^3}\mathcal{F}'\right|^2|\mathcal{D}|^{2c^{\mathrm{ext}}}\,. \tag{4.9}$$

Possible divergences. We would like to study the possible divergences in the amplitudes when we take $c^{\mathrm{ext}}\to 0$. In the subsequent computations, we take

$$Z_1 = 1\,,\qquad Z_2 = 0\,,\qquad Z_3 = Z\,,\qquad Z_4 = \infty\,, \tag{4.10}$$

and regard Z as a function of $\mathcal{T}$. In this gauge, the interaction points $z_\pm$ become

$$z_\pm = -\frac{(\alpha_1+\alpha_2)Z+\alpha_2+\alpha_3}{2\alpha_4} \pm \frac{1}{2}\frac{\alpha_1+\alpha_2}{\alpha_4}\sqrt{\left(Z+\frac{\alpha_2+\alpha_3}{\alpha_1+\alpha_2}\right)^2 + \frac{4\alpha_2\alpha_4}{(\alpha_1+\alpha_2)^2}Z}\,. \tag{4.11}$$

We write

$$z_+ - z_- = \frac{\alpha_1+\alpha_2}{\alpha_4}\sqrt{(Z-X)(Z-Y)}\,. \tag{4.12}$$

The Jacobian for the change of the variables from $\mathcal{T}$ into Z becomes

$$\frac{\partial\mathcal{T}}{\partial Z} = \frac{\alpha_4(z_+-z_-)}{Z(Z-1)}\,. \tag{4.13}$$

The amplitude (4.9) turns out to be

$$\begin{aligned}\mathcal{A} &\sim -\frac{g^2C_4}{4\pi}\int d^2Z\left|\frac{\partial\mathcal{T}}{\partial Z}\left(\mathcal{F}_{\mathrm{fin}} - \frac{c^{\mathrm{ext}}}{6(z_+-z_-)^3}\mathcal{F}'\right) + \partial_Z\mathcal{G}\right|^2|\mathcal{D}|^{2c^{\mathrm{ext}}}\\ &= -\frac{g^2C_4}{4\pi}\int d^2Z\left|\frac{\partial\mathcal{T}}{\partial Z}\right|^2|\mathcal{F}_{\mathrm{fin}}|^2|\mathcal{D}|^{2c^{\mathrm{ext}}}\\ &\quad + c^{\mathrm{ext}}\frac{g^2C_4}{4\pi}\int d^2Z\left[\frac{\partial\mathcal{T}}{\partial Z}\mathcal{F}_{\mathrm{fin}}\left(\frac{1}{6(\bar z_+-\bar z_-)^3}\frac{\partial\bar{\mathcal{T}}}{\partial\bar Z}\bar{\mathcal{F}}' + \bar{\mathcal{G}}\frac{\partial_{\bar Z}\bar{\mathcal{D}}}{\bar{\mathcal{D}}}\right)|\mathcal{D}|^{2c^{\mathrm{ext}}} + \mathrm{c.c.}\right]\\ &\quad - (c^{\mathrm{ext}})^2\frac{g^2C_4}{4\pi}\int d^2Z\left|\frac{1}{6(z_+-z_-)^3}\frac{\partial\mathcal{T}}{\partial Z}\mathcal{F}' + \mathcal{G}\frac{\partial_Z\mathcal{D}}{\mathcal{D}}\right|^2|\mathcal{D}|^{2c^{\mathrm{ext}}}\,.\end{aligned} \tag{4.14}$$

Here we have taken $c^{\rm ext}$ so that no surface terms appear in the integration by parts.

Let us analytically continue $c^{\rm ext}$ to 0. First, we consider the first term on the right hand side of eq. (4.14). Using

$$
\begin{aligned}
\frac{\partial \mathcal{T}}{\partial Z}\mathcal{F}_{\rm fin} &= \left(\frac{\partial\rho(z_+)}{\partial Z}-\frac{\partial\rho(z_-)}{\partial Z}\right)\left\langle \oint_C \frac{dz}{2\pi i}\frac{b(z)}{\partial\rho(z)}V_1^{(0)}(Z_1)V_2^{(0)}(Z_2)V_3^{(-1)}(Z)V_4^{(-1)}(Z_4)\right\rangle \\
&= \left\langle \left(\oint_C \frac{dz}{2\pi i}\frac{\partial_Z\rho(z)-\partial_Z\rho(z_-)}{\partial\rho(z)}b(z)-\oint_C \frac{dz}{2\pi i}\frac{\partial_Z\rho(z)-\partial_Z\rho(z_+)}{\partial\rho(z)}b(z)\right)\right. \\
&\qquad\qquad \left.\times V_1^{(0)}(Z_1)V_2^{(0)}(Z_2)V_3^{(-1)}(Z)V_4^{(-1)}(Z_4)\right\rangle, \qquad (4.15)
\end{aligned}
$$

and deforming the contour for the integration of b, we obtain

$$
\frac{\partial \mathcal{T}}{\partial Z}\mathcal{F}_{\rm fin} = \left\langle V_1^{(0)}(Z_1)V_2^{(0)}(Z_2)\left(e^{-\phi}V_{3L}^{\rm DDF}\right)(Z)V_4^{(-1)}(Z_4)\right\rangle, \qquad (4.16)
$$

and find that $\frac{\partial \mathcal{T}}{\partial Z}\mathcal{F}_{\rm fin}$ involves no divergences. Therefore, the first term on the right hand side of eq. (4.14) turns out to be

$$
-\frac{g^2C_4}{4\pi}\int d^2Z\left|\frac{\partial\mathcal{T}}{\partial Z}\right|^2|\mathcal{F}_{\rm fin}|^2\,, \qquad (4.17)
$$

without any problem and coincides with the amplitude of the first-quantized formalism.

In the following, we will see that the second and the third terms are vanishing as $c^{\rm ext}\to 0$. Since these terms are proportional to $c^{\rm ext}$ and $(c^{\rm ext})^2$, it is sufficient to show that the integrals in these terms do not give rise to dangerous divergences as $c^{\rm ext}\to 0$. The regions in the moduli space where these integral can be singular are as follows: (i) $Z\to Z_1,Z_2,Z_4$; (ii) $Z\to z_\pm$; (iii) $Z\to X,Y$, where X and Y are given in eq. (4.12), i.e. $z_+\to z_-$. The possible singularity in region (i) can be dealt with by the analytic continuation of the external momenta p_r. We therefore need not worry about the singularity in this case. Because of eq. (B.5), case (ii) is reduced to case (i) and need not be worried about, either. The third case is what we should study carefully. In the following, we will show that the second and the third terms on the right hand side of eq. (4.14) do not provide dangerous contributions in region (iii) and hence vanish as $c^{\rm ext}\to 0$.

Let us begin by considering the second term. We divide the integration region into three parts as

$$
\int d^2Z = \left(\int_{|Z-X|<\epsilon}+\int_{|Z-Y|<\epsilon}+\int_{|Z-X|>\epsilon \text{ and } |Z-Y|>\epsilon}\right)d^2Z\,. \qquad (4.18)
$$

The possible divergences come from the first and the second regions. Let us focus on the region $|Z-X|<\epsilon$. One can find that $\frac{\partial\mathcal{T}}{\partial Z}\mathcal{F}_{\rm fin}$ can be expanded by the Taylor series, and $\frac{\partial\mathcal{T}}{\partial Z}\frac{\mathcal{F}'}{(z_+-z_-)^3}$ and $\mathcal{G}\frac{\partial_Z\mathcal{D}}{\mathcal{D}}$ can be expanded by the Laurent series in $(Z-X)$, respectively. Taking eq. (2.16) into account, we find that the integral of each term in the expansion behaves around $Z \sim X$ as

$$\int_{|Z-X|<\epsilon} d^2 Z (Z-X)^{n-\frac{c^{\rm ext}}{96}} (\bar{Z}-\bar{X})^{m-\frac{c^{\rm ext}}{96}} \\ \sim \int_0^\epsilon dr \int_0^{2\pi} d\theta \, r^{(n+m)+1-\frac{c^{\rm ext}}{48}} e^{i\theta(n-m)} = 2\pi\delta_{n,m} \int_0^\epsilon dr \, r^{2n+1-\frac{c^{\rm ext}}{48}} \, . \qquad (4.19)$$

Here n and m are integers and at least one of them is nonnegative. This contribution is finite in the limit $c^{\rm ext} \to 0$. The same argument can be applied to the region $|Z-Y| < \epsilon$. Therefore, the second term on the right hand side of eq. (4.14) vanishes in $c^{\rm ext} \to 0$.

Let us turn to the integral of the third term on the right hand side of eq. (4.14). Due to the same discussion given above, the singular contributions are of the form

$$\int_{|Z-X|<\epsilon} d^2 Z (Z-X)^{-1-\frac{c^{\rm ext}}{96}} (\bar{Z}-\bar{X})^{-1-\frac{c^{\rm ext}}{96}} = \mathcal{O}\left(\frac{1}{c^{\rm ext}}\right) , \qquad (4.20)$$

around $Z \sim X$ for example, and we find that the possible divergence is at most $\mathcal{O}\left(\frac{1}{c^{\rm ext}}\right)$. Therefore the third term on the right hand side of eq. (4.14), obtained from such integrals multiplied by $(c^{\rm ext})^2$, is vanishing as $c^{\rm ext} \to 0$.

Thus we find that the regularization prescription works in the four-point amplitudes without counterterms and the results of the first-quantized formalism are reproduced.

5 Discussions

In this paper, we have proposed a dimensional regularization scheme to regularize the divergences caused by the colliding transverse supercurrents inserted at the interaction points of the three-string vertices of the light-cone gauge superstring field theory. We have investigated the tree amplitudes and shown that the divergences originating in the colliding supercurrents are actually regularized. We have explicitly studied the four-point amplitudes of the NS-NS closed strings, and shown that the usual results are reproduced without any counterterms in this case. Although the amplitudes without the Ramond strings have been considered, we can treat the amplitudes including them in a similar way.

There are many problems that remain to be studied. We should go on and examine the tree amplitudes with $N \geq 5$ and then the higher-loop amplitudes, and see if we need counterterms to reproduce the desired results. In these cases, there are situations in which more than two interaction points approach one another, which makes the integrations over the moduli spaces more complicated.

If it is difficult to treat the amplitudes directly, it may be better to pursue other approaches. In the dimensional regularization, we shift the space-time dimension away from the critical dimension. The light-cone gauge string field theory in such a space-time is not Lorentz invariant. However, since the light-cone gauge string field theory is a gauge fixed theory, it will be possible to find a Lorentz noninvariant but gauge invariant theory whose gauge fixed version coincides with such a noncritical string theory. If such a theory exists, the form of the possible counterterms will be severely restricted. The progress in this direction will be reported elsewhere.

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A Action of string field theory

We present the light-cone gauge string field theory of NSR closed superstrings containing the extra sector explained in section 2. Since we deal with the NS-NS sector in this paper, we consider the part involving only this sector. We define the action for the NS-NS sector as

$$S = \int dt \Bigg[\frac{1}{2} \int d1d2 \, \langle R(1,2)|\Phi\rangle_1 \left(i\frac{\partial}{\partial t} - \frac{L_0^{\mathrm{LC}(2)} + \tilde{L}_0^{\mathrm{LC}(2)} - \frac{c}{12}}{\alpha_2} \right) |\Phi\rangle_2 \\ + \frac{2g}{3} \int d1d2d3 \, \langle V_3(1,2,3)|\Phi\rangle_1 |\Phi\rangle_2 |\Phi\rangle_3 \Bigg] , \tag{A.1}$$

where t denotes the proper time, c is the central charge given in eq. (2.5), g is the coupling constant, $\alpha_r = 2p_r^+$ denotes the string-length parameter of the r-th string, and dr is the integration measure for the momentum zero modes of the r-th string,

$$dr = \frac{\alpha_r d\alpha_r}{4\pi} \frac{d^8 p_r}{(2\pi)^8} \, . \tag{A.2}$$

The string field Φ is taken to be Grassmann even and subject to

$$\int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(L_0^{\mathrm{LC}} - \tilde{L}_0^{\mathrm{LC}})} |\Phi\rangle = |\Phi\rangle \, , \quad \mathcal{P}^{\mathrm{GSO}} |\Phi\rangle = |\Phi\rangle \, , \tag{A.3}$$

where L_0^{LC} is the transverse Virasoro zero mode of the worldsheet theory of the light-cone gauge NSR superstrings containing the extra sector and $\mathcal{P}^{\mathrm{GSO}}$ denotes usual GSO projection operator for the light-cone gauge superstrings. The reflector $\langle R(1,2)|$ and the three-string vertex $\langle V_3(1,2,3)|$ are defined as

$$\begin{aligned} \langle R(1,2)| &= \delta(1,2) \; {}_2\langle 0| \, {}_1\langle 0| \, e^{-\sum_{n=1}^{\infty} \frac{1}{n} \alpha_n^{i(1)} \alpha_n^{i(2)} + i\sum_{k>0} \psi_k^{i(1)} \psi_k^{i(2)} + \text{extra sector} + \text{c.c.}} \, , \\ \langle V_3(1,2,3)| &= 4\pi\delta\left(\sum_{r=1}^{N} \alpha_r \right) \langle V_3^{\mathrm{LPP}}(1,2,3)| e^{-\Gamma^{\mathrm{LC}[3]}} \left(\partial^2\rho(z_I) \bar{\partial}^2 \bar{\rho}(\bar{z}_I) \right)^{-\frac{3}{4}} T_F^{\mathrm{LC}}(z_I) \tilde{T}_F^{\mathrm{LC}}(\bar{z}_I) \, , \\ \delta(1,2) &= (2\pi)^8 \delta^8(p_1 + p_2) \frac{4\pi}{\alpha_1} \delta(\alpha_1 + \alpha_2) \, , \\ e^{-\Gamma^{\mathrm{LC}[3]}} &= \mathrm{sgn}\,(\alpha_1\alpha_2\alpha_3) \left| \frac{e^{-2\hat{\tau}_0 \sum_{r=1}^{3} \frac{1}{\alpha_r}}}{\alpha_1\alpha_2\alpha_3} \right|^{\frac{c}{24}} \, , \qquad \hat{\tau}_0 = \sum_{r=1}^{3} \alpha_r \ln|\alpha_r| \, , \end{aligned} \tag{A.4}$$

where $\mathrm{sgn}(x)$ denotes the sign function, $\langle V_3^{\mathrm{LPP}}(1,2,3)|$ denotes the LPP vertex for three NS-NS closed strings with extra sector and T_F^{LC} is transverse supercurrent of the worldsheet theory of the light-cone gauge NSR superstring containing the extra CFT,

$$T_F^{\mathrm{LC}} = T_F^{(X,\psi)\mathrm{LC}} + T_F^{\mathrm{ext}} \, . \tag{A.5}$$

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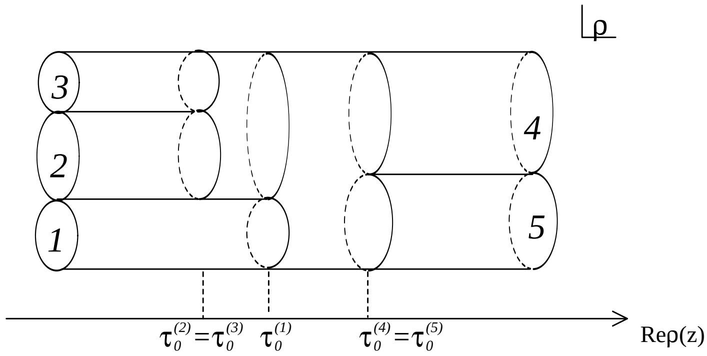

Figure 2. A typical N-string tree diagram with $N = 5$.

The Mandelstam mapping $\rho(z)$ and the interaction point z_I are defined in the next appendix.

B Mandelstam mapping

We introduce the complex coordinate ρ on the N-string tree diagram with the joining-splitting type of interaction described in figure 2. Each portion on the ρ-plane corresponding to the r-th external string ($r = 1, \dots, N$) is identified with the unit disk $|w_r| \leq 1$ of the r-th string by the relation

$$\rho = \alpha_r \ln w_r + \tau_0^{(r)} + i\beta_r \,, \tag{B.1}$$

where $\tau_0^{(r)} + i\beta_r$ is the coordinate on the ρ-plane at which the r-th string interacts (figure 2).

The N-string tree diagram can be mapped to the complex z-plane with N punctures by the Mandelstam mapping,

$$\rho(z) = \sum_{r=1}^{N} \alpha_r \ln(z - Z_r) \,, \qquad \sum_{r=1}^{N} \alpha_r = 0 \,, \tag{B.2}$$

where the point $z = Z_r$ ($r = 1, \dots, N$) is the puncture corresponding to the origin of the unit disk $w_r = 0$. The interaction points z_I are determined by $\frac{\partial\rho}{\partial z}(z_I) = 0$. These are related to $\tau_0^{(r)}$ and β_r introduced above by the equation

$$\tau_0^{(r)} + i\beta_r = \rho(z_I^{(r)}) \,, \tag{B.3}$$

where $z_I^{(r)}$ is the interaction point on the z-plane at which the r-th external string interacts.

From eq. (B.2) and the definition of the interaction points z_I, we obtain

$$\partial^2\rho(z_I) = \left(\sum_{s=1}^{N} \alpha_s Z_s\right) \frac{\prod_{J\neq I}(z_I - z_J)}{\prod_{r=1}^{N}(z_I - Z_r)} \,, \tag{B.4}$$

$$\alpha_r \prod_{s\neq r}(Z_r - Z_s) = \left(\sum_{s=1}^{N} \alpha_s Z_s\right) \prod_{I}(Z_r - z_I) \,. \tag{B.5}$$

C Derivation of $e^{-\Gamma^{\mathrm{LC}}}$

In this appendix, we will briefly explain how to derive $e^{-\Gamma^{\mathrm{LC}}}$ given in eq. (2.13).

Since the interaction vertex $\langle V_N|$ given in eq. (2.6) is defined on the ρ-plane endowed with the flat metric

$$ds^2 = d\rho d\bar{\rho}\,, \tag{C.1}$$

$e^{-\Gamma^{\mathrm{LC}}}$ is the partition function of the CFT on the ρ-plane with this metric. Here we are not dealing with the superspace formalism, and the situation is the same as that in the bosonic string case. As explained in ref. [11], its dependence on α_r and the moduli parameters $\mathcal{T}_I$ can be determined through the CFT technique by evaluating the Liouville action associated with the conformal mapping (B.2) between the ρ-plane and the z-plane with small circles around the $N-2$ interaction points z_I, the N punctures Z_r and ∞ excised. Collecting the contributions from these holes on the z-plane, we obtain

$$-\Gamma^{\mathrm{LC}} = \frac{c}{12}\sum_{r=1}^{N}\ln\left|\frac{1}{\alpha_r}\frac{\partial w_r}{\partial z}(Z_r)\right| - \frac{c}{48}\sum_I \ln\left(\partial^2\rho(z_I)\bar{\partial}^2\bar{\rho}(\bar{z}_I)\right) + \frac{c}{6}\ln\left|\lim_{z\to\infty} z^2\partial\rho(z)\right|\,, \tag{C.2}$$

up to an additive constant. Here w_r is defined in eq. (B.1). Combined with the relation

$$\ln\left(\frac{\partial w_r}{\partial z}(Z_r)\right) = -\bar{N}^{rr}_{00}\,, \tag{C.3}$$

the above equation leads to eq. (2.13), up to an overall constant, which will be discussed in the following.

Factorization of $e^{-\Gamma^{\mathrm{LC}}}$. We would like to show that $e^{-\Gamma^{\mathrm{LC}}}$ given in eq. (2.13) has the correct normalization. In the $N=3$ case, using the relations in appendix B, one can readily find that $e^{-\Gamma^{\mathrm{LC}}}$ in eq. (2.13) coincides with $e^{-\Gamma^{\mathrm{LC}[3]}}$ contained in the three-string vertex (A.4). Now that the $N=3$ case is proved, the $N\geq 4$ cases are proved if the factorization

$$\begin{aligned} e^{-\Gamma^{\mathrm{LC}[N]}}(1,2,\ldots,N) \overset{\mathcal{T}\to\infty}{\longrightarrow}\; & e^{-\Gamma^{\mathrm{LC}[N-1]}}(1,2,\ldots,N-2,m') \\ & \times\left(-e^{\frac{c}{12}\frac{\mathrm{Re}\,\mathcal{T}}{|\alpha_m|}}\right)e^{-\Gamma^{\mathrm{LC}[3]}}(m,N-1,N) \end{aligned} \tag{C.4}$$

is shown for $N\geq 4$. Here $\mathcal{T}$ is the moduli parameter of the N-string tree diagram depicted in figure 3, and $e^{-\Gamma^{\mathrm{LC}[n]}}$ denote the factor (2.13) for the n-string case. m and m' stand for the intermediate string described in figure 3 and therefore the string-length parameter α_m satisfies $\alpha_m = -\alpha_{N-1}-\alpha_N$. The factor $\left(-e^{\frac{c}{12}\frac{\mathrm{Re}\,\mathcal{T}}{|\alpha_m|}}\right)$ is the contribution from the propagator (2.4) of the lowest level state of the intermediate string.

Let us briefly explain how to show eq. (C.4). For this computation, it is useful to take $Z_N=\infty$. In this gauge, the Mandelstam mapping for the N-string case becomes

$$\rho^{[N]}(z) = \sum_{r=1}^{N-1}\alpha_r\ln(z-Z_r)\,. \tag{C.5}$$

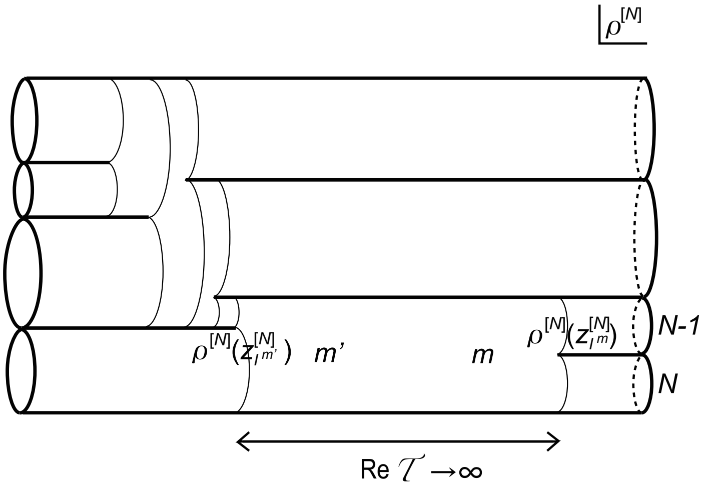

Figure 3. The interaction points for strings m and m' are denoted by $\rho^{[N]}(z^{[N]}_{I^m})$ and $\rho^{[N]}(z^{[N]}_{I^{m'}})$ respectively. We examine the limit in which $\mathcal{T} = \rho^{[N]}(z^{[N]}_{I^m}) - \rho^{[N]}(z^{[N]}_{I^{m'}}) \to \infty$.

We also introduce $\rho^{[N-1]}(z) = \sum_{r=1}^{N-2} \alpha_r \ln(z - Z_r)$. In terms of $\rho^{[N]}(z)$, $e^{-\Gamma^{\mathrm{LC}[N]}}$ is described as

$$e^{-\Gamma^{\mathrm{LC}[N]}}(1,2,\ldots,N)$$
$$= \operatorname{sgn}\left(\prod_{r=1}^{N} \alpha_r\right) \left|\frac{\alpha_N}{\prod_{r=1}^{N-1}\alpha_r}\right|^{\frac{c}{12}} \prod_I \left|\partial^2 \rho^{[N]}(z_I^{[N]})\right|^{-\frac{c}{24}} e^{-\frac{c}{12}\sum_{r=1}^{N} \mathrm{Re}\,\bar{N}_{00}^{rr[N]}}\,, \qquad \text{(C.6)}$$

where

$$\mathrm{Re}\,\bar{N}_{00}^{rr[N]} = \frac{\mathcal{T}_0^{(r)[N]}}{\alpha_r} - \sum_{s \neq r,N} \frac{\alpha_s}{\alpha_r} \ln|Z_r - Z_s|\,, \qquad \mathrm{Re}\,\bar{N}_{00}^{NN[N]} = \frac{\mathcal{T}_0^{(N)[N]}}{\alpha_N} \qquad \text{(C.7)}$$

for $r = 1, 2, \ldots, N-1$.

Let us take the limit $Z_{N-1} \to \infty$ with the other Z_r's kept fixed. In this limit, the moduli parameter $\mathcal{T}$ becomes infinity, while the other ones tend to the moduli parameters determined by the Mandelstam mapping $\rho^{[N-1]}(z)$. Taking the limit $Z_{N-1} \to \infty$ in eq. (C.6), one can find that the factorization (C.4) indeed holds.

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